A CLASSIFICATION FOR 2-ISOMETRIES OF NONCOMMUTATIVE L_p-SPACES

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ABSTRACT. In this paper we extend previous results of Banach, Lamperti and Yeadon on isometries of L_p-spaces to the non-tracial case first introduced by Haagerup. Specifically, we use operator space techniques and an extrapolation argument to prove that every 2-isometry $T\colon L_p(\mathcal{M}) \to L_p(\mathcal{N})$ between arbitrary noncommutative L_p-spaces can always be written in the form

$$T(\varphi^{\frac{1}{p}}) = w(\varphi \circ \pi^{-1} \circ E)^{\frac{1}{p}}, \qquad \varphi \in \mathcal{M}_*^+.$$

Here π is a normal *-isomorphism from $\mathcal{M}$ onto the von Neumann subalgebra $\pi(\mathcal{M})$ of $\mathcal{N}$, w is a partial isometry in $\mathcal{N}$, and E is a normal conditional expectation from $\mathcal{N}$ onto $\pi(\mathcal{M})$. As a consequence of this, any 2-isometry is automatically a complete isometry and has completely contractively complemented range.

1. Introduction

The investigation of isometries on L_p-spaces has a long tradition in the theory of Banach spaces and has connections to probability and ergodic theory. Banach [2] considered the discrete case and showed that for a surjective isometry on ℓ_p one deduces the existence of a permutation $\pi\colon \mathbb{N} \to \mathbb{N}$ and a sequence of scalars $\{\lambda_n\}$ with unit modulus so that $T(e_n) = \lambda_n e_{\pi(n)}$. Banach also stated, without proof, the form of surjective isometries of $L_p([0,1], m)$. These are **weighted composition operators**; i.e.

$$T(f)(x) = h(x)f(\varphi(x)), \qquad x \in [0,1], \tag{1.1}$$

where φ is a measurable bijection of the unit interval and h is a measurable function with $|h|^p = \frac{d(m\circ\varphi)}{dm}$. As shown by Lamperti [24], this paradigm is still basically correct for non-surjective isometries on general $L_p(X, \Sigma, \mu)$, but the

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"composition" is defined in terms of a mapping on the measurable sets (a so-called **regular set isomorphism**, see [24] or [7]). For a sufficiently nice measure space, there is still an underlying point mapping (see [12]). Notice that a set mapping is nothing but a map on the projections in the associated L_∞-algebras, so Lamperti's result can be formulated naturally in terms of von Neumann algebras! Moreover, this formulation implies that isometries must preserve **disjointness of support**, which has been a key step in every stage of the classification of L_p-isometries, including the present paper.

The most familiar noncommutative L_p-space is the Schatten-von Neumann p-ideal S_p [26], the class of operators in $\mathcal{K}(\mathfrak{H})$ (or in $\mathcal{B}(\mathfrak{H})$) whose singular values are p-summable. Segal [37] extended this concept to general semifinite von Neumann algebras. In this category, L_p-isometries have been investigated by Broise [4], Russo [36], Arazy [1], Katavolos [17], [18], [19], and Tam [46]. In 1981, Yeadon [53] finally gave a satisfactory answer.

Theorem 1. [53, Theorem 2] *Let $\mathcal{M}$ and $\mathcal{N}$ be two semifinite von Neumann algebras with traces $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$, respectively. For $1 \leq p < \infty$ and $p \neq 2$, a linear map*

$$T\colon L_p(\mathcal{M}, \tau_{\mathcal{M}}) \to L_p(\mathcal{N}, \tau_{\mathcal{N}})$$

*is an isometry if and only if there exist a normal Jordan *-monomorphism $J\colon \mathcal{M} \to \mathcal{N}$, a partial isometry $w \in \mathcal{N}$, and a positive self-adjoint operator B affiliated with $\mathcal{N}$ such that the spectral projections of B commute with $J(\mathcal{M})$, which verify the following conditions:*

(1) $w^*w = J(1) = s(B)$;
(2) $\tau_{\mathcal{M}}(x) = \tau_{\mathcal{N}}(B^p J(x))$ *for all* $x \in \mathcal{M}^+$;
(3) $T(x) = wBJ(x)$ *for all* $x \in \mathcal{M} \cap L_p(\mathcal{M}, \tau_{\mathcal{M}})$.

Moreover, J, w, and B are uniquely determined by the above conditions $(1) - (3)$.

At approximately the same time, Haagerup [9] discovered a way to construct noncommutative L_p-spaces from arbitrary von Neumann algebras. While many people worked on general noncommutative L_p-spaces over the last twenty years, until recently little progress had been made on the classification of their isometries. Watanabe wrote a series of papers (including [49] and [50]) supplying many ideas and some partial results. In [38], Watanabe's techniques are developed to obtain classification results for L_p-isometries which are complementary to those established in the present paper (by entirely different methods), and in [39] a canonical form of surjective L^p-isometries is derived, proving that $L_p(\mathcal{M})$ and $L_p(\mathcal{N})$ are isometrically isomorphic if and only if $\mathcal{M}$ and $\mathcal{N}$ are Jordan *-isomorphic.

In this paper, we use operator space techniques to give a complete classification of 2-isometries on general noncommutative L_p-spaces. We begin by extending Yeadon's result to isometries between noncommutative L_p-spaces where only the initial von Neumann algebra is assumed semifinite (using a recent result of Raynaud/Xu [34]). After some background information (section 2) this task is performed in section 3. If we drop the assumption that the initial algebra is semifinite, then $L_p(\mathcal{M}) \cap \mathcal{M} = \{0\}$, and a new kind of problem occurs. There is no longer a canonical embedding of the von Neumann algebra (or finite elements therein) into the noncommutative L_p-space, and therefore we may no longer use the lattice of projections canonically. This has been the main obstruction for applying Yeadon's groundbreaking technique [53]. We found that operator space methods can be employed to overcome this difficulty. The operator space structure of noncommutative L_p-spaces was introduced by Pisier [30] and later extended to the non-semifinite case (see [14] and [31], the structure in [6] is not entirely compatible here). In section 4 we are essentially interested in taking columns and rows of elements in L_p in order to classify **complete** isometries. Previous work for the case $p = 1$ was done by [28] and [27]. While isometries are connected with Jordan structure, a 2-isometry must already preserve multiplicative structure. This is one of the key observations for our result, the other being an extrapolation argument which shows that if we have an L_p-isometry for one $p \neq 2$, then we have an associated isometry for any other p. The case $p = 4$ connects with the theory of selfpolar forms (from [10]) and thus can be used to construct a conditional expectation, leading to a proof of the main theorems. The final section of the paper consists of remarks.

To be more specific let us fix some notation. If $\mathcal{N}$ is a von Neumann algebra, then $L_p(\mathcal{N})$ is the linear span of elements $\varphi^{\frac{1}{p}}$, where φ ranges over the positive states in $\mathcal{N}_*$.

Theorem 2. *Let $1 \leq p \neq 2 < \infty$ and $T\colon L_p(\mathcal{M}) \to L_p(\mathcal{N})$ be a linear map. The following are equivalent:*

i) *$id \otimes T\colon L_p(M_2 \otimes \mathcal{M}) \to L_p(M_2 \otimes \mathcal{N})$ is an isometry,*

ii) *$id \otimes T\colon L_p(M_m \otimes \mathcal{M}) \to L_p(M_m \otimes \mathcal{N})$ is an isometry for all $m \in \mathbb{N}$,*

iii) *There exists an injective normal *-homomorphism $\pi\colon \mathcal{M} \to \mathcal{N}$, a partial isometry $w \in \mathcal{N}$, and a conditional expectation E from $\mathcal{N}$ onto $\pi(\mathcal{M})$ such that*

$$T(\varphi^{\frac{1}{p}}) \;=\; w(\varphi \circ \pi^{-1} \circ E)^{\frac{1}{p}}, \qquad \varphi \in \mathcal{M}_*^+ \,. \tag{1.2}$$

In (1.2) we have to use the fact that a map defined on positive vectors extends uniquely to the whole of $L_p(\mathcal{N})$. Moreover, we recall that $L_p(\mathcal{N})$ is an $\mathcal{N} - \mathcal{N}$ bimodule, so that $w\varphi^{\frac{1}{p}}$ is a well-defined element satisfying $\|w\varphi^{\frac{1}{p}}\|_p = \varphi(1)^{\frac{1}{p}}$ (see [47] for more details). In the σ-finite case, $\phi^{\frac{1}{p}}\mathcal{N}$ is dense in $L_p(\mathcal{N})$ for every normal faithful state ϕ. Then we may alternatively describe T by

$$T(\phi^{\frac{1}{p}}x) \;=\; w(\phi \circ \pi^{-1} \circ E)^{\frac{1}{p}}\pi(x), \qquad x \in \mathcal{M}. \tag{1.3}$$

Notice that when ϕ is not tracial, the inclusion mapping $i_p\colon \mathcal{N} \to L_p(\mathcal{N})$, $i_p(x) = \phi^{\frac{1}{p}}x$, no longer preserves disjoint left supports. This is the obstacle mentioned above.

To end this introduction let us mention some recent sources which will provide further background to interested readers. The book of Fleming and Jamison [7] is an up-to-date survey on isometries in Banach spaces (including classical L_p-spaces); their companion volume will treat the case of noncommutative L_p-spaces. The handbook article of Pisier and Xu [32] provides a general overview of noncommutative L_p-spaces, focusing on Banach space properties and including a rich bibliography. There is also recent groundbreaking work on the **non-isometric** isomorphism/embedding question by Haagerup, Rosenthal, and Sukochev [11] which is entirely disjoint to the isometric analysis of this paper.

2. Some background

In this section we provide some background on noncommutative L_p-spaces and their operator space structure. The readers are referred to Nelson [25], Kosaki [22], Terp [47] and [48] for details on noncommutative L_p-spaces and to Pisier [30], [31], Fidaleo [6], and Junge-Ruan-Xu [14] for the canonical operator space structure on these spaces. Pisier and Xu's recent survey paper [32] provides a nice overview of this subject. For general background on the modular theory of von Neumann algebras, we recommend the Takesaki *oeuvre* [43], [44] and [45], and Strătilă [41].

We first recall the L_p-space ($1 \leq p < \infty$) associated with a semifinite algebra $\mathcal{M}$ equipped with a given normal faithful semifinite trace τ (simply called a "trace" from here on). Consider the set

$$\{T \in \mathcal{M} \mid \|T\|_p \triangleq \tau(|T|^p)^{1/p} < \infty\}.$$

It can be shown that $\|\cdot\|_p$ defines a norm on this set. Then the norm completion, which is denoted $L_p(\mathcal{M},\tau)$, is the noncommutative L_p-space obtained from $(\mathcal{M},\tau)$. It turns out that one can identify elements of $L_p(\mathcal{M},\tau)$ with certain τ-measurable operators affiliated with $\mathcal{M}$ (see [25]). Clearly τ is playing the role of integration here.

A von Neumann algebra lacking a faithful trace is not amenable to the previous definition. There are several alternative constructions which work in full generality. Let us first recall the construction initiated by Haagerup [9] and carried out in detail by Terp [47]. Choose a normal faithful semifinite weight ϕ on $\mathcal{M}$. We consider the one-parameter modular automorphism σ_t^ϕ (associated with ϕ) on $\mathcal{M}$ and obtain a semifinite von Neumann algebra $\widetilde{\mathcal{M}} \triangleq \mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$ which has an induced trace τ and a trace-scaling dual action θ such that $\tau \circ \theta_s = e^{-s}\tau$ for all $s \in \mathbb{R}$.

The original von Neumann algebra $\mathcal{M}$ can be identified with a θ-invariant von Neumann subalgebra $L_\infty(\mathcal{M})$ of $\widetilde{\mathcal{M}}$. For $1 \leq p < \infty$, the noncommutative L_p-space $L_p(\mathcal{M},\phi)$ is defined to be the space of all (unbounded) τ-measurable operators affiliated with $\widetilde{\mathcal{M}}$ such that $\theta_s(T) = e^{-\frac{s}{p}}T$ for all $s \in \mathbb{R}$. It is known from Terp [47, Chapter II] that there is a one-to-one correspondence between bounded (positive) linear functionals $\psi \in \mathcal{M}_*$ and τ-measurable (positive self-adjoint) operators $h_\psi \in L_1(\mathcal{M},\phi)$ under the connection given by

$$\widehat{\psi}(\tilde{x}) = \tau(h_\psi \tilde{x}), \qquad \tilde{x} \in \widetilde{\mathcal{M}},$$

where $\widehat{\psi}$ is the so-called **dual weight** for ψ. This correspondence actually extends to all of $\mathcal{M}_*$ and $L_1(\mathcal{M},\phi)$, and we may define the "tracial" linear functional

$tr = tr_{\mathcal{M}}$: $L_1(\mathcal{M}, \phi) \to \mathbb{C}$ by
(2.1)
$$tr(h_\psi) = \psi(1), \quad \text{satisfying} \quad tr(h_\psi x) = tr(x h_\psi) = \psi(x), \quad \psi \in \mathcal{M}_*^+,\ x \in \mathcal{M}.$$

Given any $h \in L_p(\mathcal{M})$, we have the polar decomposition $h = w|h|$, where $|h|$ is a positive operator in $L_p(\mathcal{M})^+$ and w is a partial isometry contained in $\mathcal{M}$ such that the projection $s_l(h) = ww^*$ is the *left support* of h and the projection $s_r(h) = w^*w$ is the *right support* of h. (We simply use s for the support of positive vectors, operators, and maps.) We can define a Banach space norm on $L_p(\mathcal{M}, \phi)$ by

$$\|h\|_p = tr(|h|^p)^{\frac{1}{p}} = \psi(1)^{\frac{1}{p}} \tag{2.2}$$

if $\psi \in \mathcal{M}_*$ corresponds to $|h|^p \in L_1(\mathcal{M}, \phi)^+$. With this norm, it is easy to see that $L_1(\mathcal{M}, \phi)$ is isometrically and orderly isomorphic to $\mathcal{M}_*$. We note that up to isometry the noncommutative L_p-space constructed above is actually independent of the choice of normal faithful semifinite weight on $\mathcal{M}$. Therefore, we will simply write $L_p(\mathcal{M})$ if there is no confusion.

We note that for any positive operator $h \in L_p(\mathcal{M})^+$, h^p is a positive operator in $L_1(\mathcal{M})^+$ and thus we can write $h^p = h_\psi$ for a corresponding positive linear functional $\psi \in \mathcal{M}_*^+$. Therefore, we may identify h with $\psi^{\frac{1}{p}}$, i.e. we can simply write

$$h = \psi^{\frac{1}{p}}.$$

This notation, discussed specifically in [52], [5, Section V.B.α] and [40], provides the relations

$$\psi^{it}\varphi^{-it} = (D\psi : D\varphi)_t, \qquad \varphi^{it} x \varphi^{-it} = \sigma_t^\varphi(x), \qquad \varphi, \psi \in \mathcal{M}_*^+,\ \varphi \text{ faithful}.$$

It will be used in section 4, where it suggests that the main results really deal with "noncommutative weighted composition operators".

For $1 \le p < \infty$, we let $L_p(\mathcal{M})'$ denote the dual space of $L_p(\mathcal{M})$. Then we can obtain the isometric isomorphism $L_p(\mathcal{M})' = L_{p'}(\mathcal{M})$ under the *trace duality*

$$\langle x, y \rangle = tr(xy) = tr(yx) \tag{2.3}$$

for all $x \in L_p(\mathcal{M})$ and $y \in L_{p'}(\mathcal{M})$. (Throughout p' denotes the conjugate exponent of p; i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.)

One of the advantages in using Haagerup's approach is that the natural $\mathcal{M}-\mathcal{M}$ bimodule structure on $L_p(\mathcal{M})$ is just operator composition. It was shown in [15, Lemma 1.2] that for any $h \in L_p(\mathcal{M})$, we have

$$\overline{\{xh \mid x \in \mathcal{M}\}} = L_p(\mathcal{M})s_r(h) \text{ and } \overline{\{hx \mid x \in \mathcal{M}\}} = s_l(h)L_p(\mathcal{M}). \tag{2.4}$$

In particular, if $\mathcal{M}$ is a σ-finite von Neumann algebra and ϕ is a normal faithful positive linear functional on $\mathcal{M}$, then $h = \phi^{\frac{1}{p}}$ is a *cyclic vector* in $L_p(\mathcal{M})$, i.e.

$$\{x\phi^{\frac{1}{p}} \mid x \in \mathcal{M}\} \text{ and } \{\phi^{\frac{1}{p}}x \mid x \in \mathcal{M}\}$$

are norm dense in $L_p(\mathcal{M})$. In this case, we can obtain the following result (see Junge and Sherman [15, Lemma 1.3]).

Lemma 2.1. *Let $\phi \in \mathcal{M}_*^+$ be faithful. A bounded net $\{x_\alpha\} \in \mathcal{M}$ converges strongly to x, $x_\alpha \xrightarrow{s} x$, if and only if $x_\alpha\phi^{\frac{1}{p}} \to x\phi^{\frac{1}{p}}$ (or $\phi^{\frac{1}{p}}x_\alpha \to \phi^{\frac{1}{p}}x$) in $L_p(\mathcal{M})$.*

Next let us recall Kosaki's complex interpolation construction of $L^p(\mathcal{M})$ (see [22]). We assume that $\mathcal{M}$ is a σ-finite von Neumann algebra and ϕ is a normal faithful state on $\mathcal{M}$. Then we may identify $\mathcal{M}$ with a subspace of $\mathcal{M}_*$ by the *right embedding*

$$x \in \mathcal{M} \mapsto \phi x \in \mathcal{M}_*,$$

where $\langle \phi x, y\rangle = \phi(xy)$. We use the right embedding (instead of the left embedding) in this paper for the convenience of our notation in representation theorems. Then the complex interpolation $[\mathcal{M}, \mathcal{M}_*]_{\frac{1}{p}}$ is a Banach space which can be isometrically identified with $\phi^{\frac{1}{p'}}L_p(\mathcal{M})$ by

$$\left\|\phi^{\frac{1}{p'}}h\right\|_{[\mathcal{M},\mathcal{M}_*]_{\frac{1}{p}}} = \|h\|_{L_p(\mathcal{M})}$$

for all $h \in L_p(\mathcal{M})$ (see [22, Theorem 9.1]). In particular, any $\phi^{\frac{1}{p}}x \in L_p(\mathcal{M})$ corresponds to an element ϕx in $[\mathcal{M}, \mathcal{M}_*]_{\frac{1}{p}}$ since

$$\phi x = \phi^{\frac{1}{p'}}(\phi^{\frac{1}{p}}x). \tag{2.5}$$

In this case, we have

$$\|\phi x\|_{[\mathcal{M},\mathcal{M}_*]_{\frac{1}{p}}} = \left\|\phi^{\frac{1}{p}}x\right\|_{L_p(\mathcal{M})}. \tag{2.6}$$

We will simply write $L_p(\mathcal{M}) = [\mathcal{M}, \mathcal{M}_*]_{\frac{1}{p}}$ when there is no confusion.

Using the complex interpolation, Pisier [30] contructed a canonical operator space matrix norm

$$M_n(L_p(\mathcal{M})) = [M_n(\mathcal{M}), M_n(\mathcal{M}_*^{op})]_{\frac{1}{p}} \tag{2.7}$$

on $L_p(\mathcal{M})$ (also see [31] and [14]). For each $n \in \mathbb{N}$, we may also consider the noncommutative S_p^n-integral $S_p^n[L_p(\mathcal{M})]$ of $L_p(\mathcal{M})$ and it turns out that we have the isometric isomorphism

$$S_p^n[L_p(\mathcal{M})] = L_p(M_n \bar{\otimes} \mathcal{M}). \tag{2.8}$$

Moreover we can recover the canonical matrix norm on $L_p(\mathcal{M})$ by

$$\|x\|_{M_n(L_p(\mathcal{M}))} = \sup\{\|\alpha x \beta\|_{S_p^n[L_p(\mathcal{M})]} : \|\alpha\|_{S_{2p}^n}, \|\beta\|_{S_{2p}^n} \leq 1\}. \tag{2.9}$$

Consequently, a linear map $T\colon L_p(\mathcal{M}) \to L_p(\mathcal{N})$ is a complete contraction (respectively, a complete isometry) if and only if for every $n \in \mathbb{N}$,

$$\mathrm{id}_{S_p^n} \otimes T\colon S_p^n[L_p(\mathcal{M})] \to S_p^n[L_p(\mathcal{N})]$$

is a contraction (respectively, an isometry).

Finally let us recall the following equality condition for the noncommutative Clarkson inequality, which was shown for semifinite von Neumann algebras by Yeadon [53], for general von Neumann algebras with $2 < p < \infty$ by Kosaki [21], and just recently for all $p \neq 2$ by Raynaud and Xu [34].

Theorem 2.2. *For $h, k \in L_p(\mathcal{M})$, $0 < p \neq 2 < \infty$,*

$$\|h + k\|_p^p + \|h - k\|_p^p = 2(\|h\|_p^p + \|k\|_p^p) \iff hk^* = h^*k = 0. \tag{2.10}$$

Let us agree to say that L_p vectors h and k are *orthogonal* when they satisfy the conditions in (2.10). We mentioned earlier that isometries on classical L_p-spaces preserve disjointness of support. Theorem 2.2 tells us that isometries on general noncommutative L_p-spaces preserve orthogonality. This plays a key role in several of our proofs.

3. A generalization of Yeadon's theorem

In 1981, Yeadon obtained a very satisfactory and complete description for isometries between noncommutative L_p-spaces associated with semifinite von Neumann algebras (Theorem 1 above). We found that an analog of Yeadon's result still holds if the initial algebra $\mathcal{M}$ is semifinite and the range algebra $\mathcal{N}$ is arbitrary. The proof of this generalized result is almost the same as that given in Yeadon [53]. The major difference is that if $\mathcal{N}$ is a general von Neumann algebra with a normal faithful semifinite weight ϕ, the positive self-adjoint operator B is affiliated with the semifinite von Neumann algebra crossed product $\widetilde{\mathcal{N}} = \mathcal{N} \rtimes_{\sigma^\phi} \mathbb{R}$ (instead of $\mathcal{N}$), and condition (2) in Theorem 1 should be revised as

$$\tau_{\mathcal{M}}(x) = tr_{\mathcal{N}}(B^p J(x))$$

for all $x \in \mathcal{M}^+$, where $tr_{\mathcal{N}}$ is the Haagerup trace introduced in (2.1). We will outline the proof of this generalized result in the following theorem since it will provide us necessary notations and motivation for the rest of the paper. We note that a proof by different methods can be found in Sherman [38]. Yeadon could not have proven such a result because general L_p-spaces were only being invented as he was writing his paper, but he **did** prove it for preduals (the $p = 1$ case) in [53, §4].

In the rest of this section, let us assume that $\mathcal{M}$ is a semifinite von Neumann algebra with a (normal faithful semifinite) trace $\tau_{\mathcal{M}}$ and $\mathcal{N}$ is an arbitrary von Neumann algebra with a normal faithful semifinite weight ϕ, which induces the normal faithful semifnite trace $\tau_{\widetilde{\mathcal{N}}}$ on the crossed product $\widetilde{\mathcal{N}} = \mathcal{N} \rtimes_{\sigma^\phi} \mathbb{R}$, and the Haagerup trace $tr_{\mathcal{N}}$ on $L_1(\mathcal{N})$.

Theorem 3.1. *For $1 \leq p < \infty$ and $p \neq 2$, a linear map*

$$T\colon L_p(\mathcal{M}, \tau_{\mathcal{M}}) \to L_p(\mathcal{N})$$

*is an isometry if and only if there exist a normal Jordan *-monomorphism $J\colon \mathcal{M} \to \mathcal{N}$, a partial isometry $w \in \mathcal{N}$, and a positive self-adjoint operator B affiliated with $\widetilde{\mathcal{N}}$ such that $\theta_s(B) = e^{-\frac{s}{p}} B$ for all $s \in \mathbb{R}$ and the spectral projections of B commute with $J(\mathcal{M}) \subset \mathcal{N} \subset \widetilde{\mathcal{N}}$, which verify the following conditions:*

(1) *$w^* w = J(1) = s(B)$;*
(2) *$\tau_{\mathcal{M}}(x) = tr_{\mathcal{N}}(B^p J(x))$ for all $x \in \mathcal{M} \cap L_p(\mathcal{M})$;*
(3) *$T(x) = wBJ(x)$ for all $x \in \mathcal{M} \cap L^p(\mathcal{M}, \tau_{\mathcal{M}})$.*

Moreover, J, w, and B are uniquely determined by the above conditions $(1)-(3)$.

Proof. First note that the stated conditions do define an isometry; (2) implies

$$\|x\|_p^p = \tau_{\mathcal{M}}(|x|^p) = tr_{\mathcal{N}}(B^p J(|x|^p)) = tr_{\mathcal{N}}(|BJ(x)|^p) = \|T(x)\|_p^p$$

for any $x \in \mathcal{M} \bigcap L_p(\mathcal{M})$. In the rest of the proof we derive these conditions for an arbitrary isometry T.

For each $\tau_{\mathcal{M}}$-finite projection $e \in \mathcal{M}$, we let $T(e) = w_e B_e$ be the polar decomposition of $T(e) \in L_p(\mathcal{N})$. Then w_e is a partial isometry in $\mathcal{N}$ and $B_e = |T(e)|$

is a positive element of $L_p(\mathcal{N})$ such that

$$B_e w_e^* w_e = B_e = w_e^* w_e B_e. \tag{3.1}$$

If we define $J(e) = w_e^* w_e = s_r(T(e)) = s(B_e)$ to be the corresponding projection in $\mathcal{N}$, then B_e commutes with $J(e)$.

Let e and f be two mutually orthogonal $\tau_{\mathcal{M}}$-finite projections in $\mathcal{M}$. Since T is an isometry, we have

$$\|T(e) \pm T(f)\|_p^p = \|T(e \pm f)\|_p^p = \|e \pm f\|_p^p = \|e\|_p^p + \|f\|_p^p = \|T(e)\|_p^p + \|T(f)\|_p^p.$$

Then the Clarkson inequality is an equality, and we can conclude by Theorem 2.2 that

$$T(e)^* T(f) = T(e)T(f)^* = 0.$$

This implies that

$$B_e B_f = 0, \qquad J(e)J(f) = 0, \qquad \text{and} \qquad w_e^* w_f = w_e w_f^* = 0.$$

The linearity of T gives $w_{e+f} B_{e+f} = w_e B_e + w_f B_f$, from which

$$\begin{aligned} B_{e+f}^2 &= (w_{e+f} B_{e+f})^* (w_{e+f} B_{e+f}) = (w_e B_e + w_f B_f)^* (w_e B_e + w_f B_f) \\ &= B_e^2 + B_f^2 = (B_e + B_f)^2 \end{aligned}$$

and so

$$w_e B_e + w_f B_f = w_{e+f} B_{e+f} = w_{e+f} B_e + w_{e+f} B_f.$$

This implies

$$B_{e+f} = B_e + B_f, \quad J(e+f) = J(e) + J(f), \quad \text{and} \quad w_{e+f} = w_e + w_f. \tag{3.2}$$

If $x = \sum \lambda_i e_i \in \mathcal{M}$ is a self-adjoint simple operator with mutually orthogonal τ-finite projections $\{e_i\}$ in $\mathcal{M}$, we define $J(x) = \sum \lambda_i J(e_i)$. It is easy to verify that

$$J(x^2) = J(x)^2 \text{ and } \|J(x)\|_\infty = \|x\|_\infty.$$

Moreover, we have $J(\lambda x) = \lambda x$ for all real λ and $J(x+y) = J(x) + J(y)$ if x and y are commuting self-adjoint simple operators in $\mathcal{M}$. In general, for any self-adjoint operator $x \in \mathcal{M}$, there exists a sequence of simple functions f_n on the spectrum of x with $f_n(0) = 0$ which converges uniformly to the identity function $f(\lambda) = \lambda$. We define $J(x)$ to be the $\|\cdot\|_\infty$-limit of $J(f_n(x))$ in $\mathcal{N}$.

Now fix a $\tau_{\mathcal{M}}$-finite projection $e \in \mathcal{M}$. If f is a projection in $\mathcal{M}$ such that $f \leq e$, then $T(f) = T(e)J(f)$ and thus

$$T(x) = T(e)J(x) = w_e B_e J(x) \tag{3.3}$$

for all $x \in e\mathcal{M}_{sa}e$. It follows that for any $x, y \in e\mathcal{M}_{sa}e$,

$$T(e)(J(x+y) - J(x) - J(y)) = T(x+y) - T(x) - T(y) = 0.$$

This implies

$$J(x+y) - J(x) - J(y) = 0$$

and thus J is a real linear map from $e\mathcal{M}_{sa}e$ into $\mathcal{N}_{sa}$. Next we can extend J to a *-linear map on all of $e\mathcal{M}e$ by

$$J(x+iy) = J(x) + iJ(y), \qquad \forall x, y \in e\mathcal{M}_{sa}e.$$

Still on $e\mathcal{M}e$, we have

$$\begin{aligned} J[(x+iy)^2] &= J[x^2 - y^2 + i((x+y)^2 - x^2 - y^2)] \\ &= J(x)^2 - J(y)^2 + i((J(x)+J(y))^2 - J(x)^2 - J(y)^2) \\ &= [J(x+iy)]^2. \end{aligned}$$

This shows that J is a Jordan *-monomorphism on $e\mathcal{M}e$. The normality of J follows from (3.3) and Lemma 2.1. It is clear from (3.1) and the above construction that all spectral projections of B_e commute with $J(e\mathcal{M}e)$.

If $\tau_{\mathcal{M}}$ is a finite trace, we may take $B = B_1$ and the theorem is proved. If $\tau_{\mathcal{M}}$ is not finite, some gluing must be done. In this case, we may assume that $\{e_\alpha\}$ is the (increasing) net of all $\tau_{\mathcal{M}}$-finite projections in $\mathcal{M}$. Then $e_\alpha \to 1$ in the strong operator topology. By (3.2) and (3.3), we may find a partial isometry $w \in \mathcal{N}$ (as a strong limit of $\{w_{e_\alpha}\}$) and a positive self-adjoint operator B (as the supremum of $\{B_{e_\alpha}\}$) affiliated with $\widetilde{\mathcal{N}}$, and extend J to a normal Jordan *-monomorphism from $\mathcal{M}$ into $\mathcal{N}$ such that conditions (1) – (3) are satisfied. In this case, the spectral projections of B commute with $J(\mathcal{M})$, and B satisfies $\theta_s(B) = e^{-\frac{s}{p}}B$ for all $s \in \mathbb{R}$. We also have

$$w_{e_\alpha} = wJ(e_\alpha), \quad B_{e_\alpha} = BJ(e_\alpha), \tag{3.4}$$

and

$$tr_{\mathcal{N}}(B^p y) = \sup\{tr_{\mathcal{N}}(B^p_{e_\alpha} y)\} \tag{3.5}$$

for every $y \in \mathcal{N}^+$. Since each $B^p_{e_\alpha} \in L_1(\mathcal{N})$ corresponds to a normal positive linear functional $\varphi_\alpha = tr_{\mathcal{N}}(B^p_{e_\alpha}\cdot)$ on $\mathcal{N}$, B^p corresponds to a normal semifinite weight $\varphi = tr_{\mathcal{N}}(B^p\cdot)$ on $\mathcal{N}$, which has support $J(1)$. Therefore, φ is faithful when restricted to $J(1)\mathcal{N}J(1)$. The uniqueness of J, w, and B is an easy consequence which we leave to the reader. □

If we assume $J(1) = 1$ then φ is a normal faithful semifinite weight on $\mathcal{N}$. In this case, we may write $L_p(\mathcal{N}) = L_p(\mathcal{N}, \varphi)$ and it follows from the conditions (2) and (3) in Theorem 3.2 that the Jordan *-monomorphism $J\colon \mathcal{M} \to \mathcal{N}$ induces an orderly isometric injection J_p^φ from $L_p(\mathcal{M}, \tau_\mathcal{M})$ into $L_p(\mathcal{N}, \varphi)$, which is given by

$$J_p^\varphi(x) = BJ(x) \tag{3.6}$$

for all $x \in \mathcal{M} \bigcap L_p(\mathcal{M}, \tau)$. If $J(1) \neq 1$, then we may orderly and isometrically identify $L_p(\mathcal{M}, \tau_\mathcal{M})$ with a subspace of $L_p(J(1)\mathcal{N}J(1), \varphi) \simeq J(1)L_p(\mathcal{N})J(1)$.

Using a standard matricial technique, we obtain the following equivalence result. The $p = 1$ case is mentioned (without proof) in [27].

Proposition 3.2. *Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $\mathcal{N}$ be an arbitrary von Neumann algebra. Let $1 \leq p < \infty$ and $p \neq 2$. For an isometry $T = wBJ\colon L_p(\mathcal{M}, \tau_\mathcal{M}) \to L_p(\mathcal{N})$, the following statements are equivalent:*

(i) *T is a complete isometry,*
(ii) *T is a 2-isometry,*
(iii) *the Jordan map $J\colon \mathcal{M} \to \mathcal{N}$ is multiplicative.*

Proof. It is obvious that (i) $\Rightarrow$ (ii).

If J is multiplicative from $\mathcal{M}$ into $\mathcal{N}$ then $J_n = \mathrm{id}_n \otimes J$ is a *-isomorphism from $M_n(\mathcal{M})$ into $M_n(\mathcal{N})$ for every $n \in \mathbb{N}$. In this case, we can write

$$id_{S_p^n} \otimes T : S_p^n[L_p(\mathcal{M})] = L_p(M_n \bar{\otimes} \mathcal{M}, \mathrm{tr}_n \otimes \tau_\mathcal{M}) \to S_p^n[L_p(\mathcal{N})] = L_p(M_n \bar{\otimes} \mathcal{N})$$

as

$$id_{S_p^n} \otimes T = w_n B_n J_n$$

Here $w_n = I_n \otimes w$ is a partial isometry in $M_n(\mathcal{N})$ such that $w_n^* w_n = J_n(1_n)$. If we let $B = \sup\{B_{e_\alpha}\}$ as in the proof of Theorem 3.1, then the positive selfadjoint operator $B_n = \sup_\alpha(B_{e_\alpha} \oplus_p B_{e_\alpha} \oplus_p \cdots \oplus_p B_{e_\alpha})$ is affiliated with $M_n(\widetilde{\mathcal{N}})$ whose spectral projections commute with $J_n(M_n(\mathcal{M}))$. We have that

$$(\mathrm{tr}_n \otimes \tau_\mathcal{M})(x) = (\mathrm{tr}_n \otimes tr_\mathcal{N})(B_n^p J_n(x))$$

for all $x \in M_n(\mathcal{M})^+$. Then each $id_{S_p^n} \otimes T\colon S_p^n[L_p(\mathcal{M})] \to S_p^n[L_p(\mathcal{N})]$ is an isometry by Theorem 3.1. Therefore, T is a complete isometry and we proved (iii) $\Rightarrow$ (i).

It remains to prove (ii) implies (iii). First T is an isometry and thus has the representation $T = wBJ$ by Theorem 3.1. Since T is also a 2-isometry, we deduce an isometry

$$\widetilde{T} = id_{S_p^2} \otimes T \colon S_p^2[L_p(\mathcal{M}, \tau_{\mathcal{M}})] \to S_p^2[L_p(\mathcal{N})].$$

Applying Theorem 3.1 to $\widetilde{T}$, we obtain a normal Jordan *-monomorphism

$$\tilde{J} \colon M_2 \bar{\otimes} \mathcal{M} \to M_2 \bar{\otimes} \mathcal{N},$$

a partial isometry $\widetilde{w} \in M_2 \bar{\otimes} \mathcal{N}$, and a positive selfadjoint operator $\tilde{B}$ satisfying the conditions in Theorem 3.1.

If e_i $(i = 1, 2)$ are $\tau_{\mathcal{M}}$-finite projections in $\mathcal{M}$, then $\tilde{e} = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$ is a $(\mathrm{tr}_2 \otimes \tau_{\mathcal{M}})$-finite projection in $M_2 \bar{\otimes} \mathcal{M}$. Let $\widetilde{T}(\tilde{e}) = w_{\tilde{e}} B_{\tilde{e}}$ be the polar decomposition of $T_2(\tilde{e})$ and let $T(e_i) = w_{e_i} B_{e_i}$ be the polar decomposition of $T(e_i)$ $(i = 1, 2)$. Since

$$\widetilde{T}(\tilde{e}) = \begin{bmatrix} T(e_1) & 0 \\ 0 & T(e_2) \end{bmatrix} = \begin{bmatrix} w_{e_1} & 0 \\ 0 & w_{e_2} \end{bmatrix} \begin{bmatrix} B_{e_1} & 0 \\ 0 & B_{e_2} \end{bmatrix}$$

we must have, by the uniqueness of the polar decomposition,

$$w_{\tilde{e}} = \begin{bmatrix} w_{e_1} & 0 \\ 0 & w_{e_2} \end{bmatrix} \text{ and } B_{\tilde{e}} = \begin{bmatrix} B_{e_1} & 0 \\ 0 & B_{e_2} \end{bmatrix}.$$

According to the definition of $\widetilde{J}$ given in the proof of Theorem 3.1, we have

$$\widetilde{J}\left(\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \right) = \begin{bmatrix} J(e_1) & 0 \\ 0 & J(e_2) \end{bmatrix},$$

and thus we can conclude that

$$\widetilde{J}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right) = \begin{bmatrix} J(x) & 0 \\ 0 & J(y) \end{bmatrix}$$

for all $x, y \in \mathcal{M}$. Since $\widetilde{T}$ is an M_2-bimodule morphism, we can write

$$\widetilde{T}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) = \widetilde{T}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} T(x) & 0 \\ 0 & T(y) \end{bmatrix}.$$

Then for any $x, y \in \mathcal{M}$, we obtain

$$\widetilde{J}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & J(x) \\ J(y) & 0 \end{bmatrix}.$$

From this we can conclude that $J(xy) = J(x)J(y)$ since

$$\begin{aligned} \begin{bmatrix} J(xy) & 0 \\ 0 & J(yx) \end{bmatrix} &= \widetilde{J}\left(\begin{bmatrix} xy & 0 \\ 0 & yx \end{bmatrix} \right) = \widetilde{J}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}^2 \right) = \widetilde{J}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right)^2 \\ &= \begin{bmatrix} 0 & J(x) \\ J(y) & 0 \end{bmatrix} \begin{bmatrix} 0 & J(x) \\ J(y) & 0 \end{bmatrix} = \begin{bmatrix} J(x)J(y) & 0 \\ 0 & J(y)J(x) \end{bmatrix}. \end{aligned}$$

Therefore J is multiplicative. Finally we note that we can conclude from the above calculations that $\widetilde{J} = J_2$, $\widetilde{w} = w_2$ and $\widetilde{B} = B_2$. □

If $T = wBJ$: $L_p(\mathcal{M}, \tau_{\mathcal{M}}) \to L_p(\mathcal{N})$ is a 2-isometry (or equivalently, a complete isometry) with J a normal *-monomorphism from $\mathcal{M}$ into $\mathcal{N}$, then

$$S = w^*T = BJ \tag{3.7}$$

is a completely positive and completely isometric injection from $L_p(\mathcal{M}, \tau_{\mathcal{M}})$ into $L_p(\mathcal{N})$. Therefore, we can completely orderly and completely isometrically identify $L_p(\mathcal{M}, \tau_{\mathcal{M}})$ with the operator subspace $S(L_p(\mathcal{M}, \tau_{\mathcal{M}}))$ of $L_p(\mathcal{N})$.

Proposition 3.3. *Let T: $L_p(\mathcal{M}, \tau_{\mathcal{M}}) \to L_p(\mathcal{N})$ be a 2-isometry (or equivalently, a complete isometry). Then $T(L_p(\mathcal{M}, \tau_{\mathcal{M}}))$ is completely contractively complemented in $L_p(\mathcal{N})$.*

If, in addition, T is positive, then $T(L_p(\mathcal{M}, \tau_{\mathcal{M}}))$ is completely positively and completely contractively complemented in $L_p(\mathcal{N})$.

Proof. Assume the notation of Theorem 3.1. We have that J is multiplicative by Proposition 3.2, so that $J(\mathcal{M})$ is a von Neumann subalgebra of $\mathcal{N}$. Without loss of generality, we may assume that $\varphi = tr_{\mathcal{N}}(B^p \cdot)$ is faithful on $\mathcal{N}$. Otherwise we restrict our argument to $J(1)\mathcal{N}J(1)$.

Since B^p is the Pedersen-Takesaki derivative [29] of $\widehat{\varphi}$ on $\widetilde{\mathcal{N}}$ with respect to $\tau_{\widetilde{\mathcal{N}}}$, we have that

$$\sigma_t^{\varphi}(y) = \sigma_t^{\widehat{\varphi}}(y) = B^{ipt} y B^{-ipt}, \qquad y \in \mathcal{N} \subset \widetilde{\mathcal{N}}.$$

In particular, we have

$$\sigma_t^{\varphi}(J(x)) = B^{ipt} J(x) B^{-ipt} = J(x)$$

for all $x \in \mathcal{M}$, since B commutes with $J(\mathcal{M})$. Then we can conclude from Takesaki's theorem [42] that there exists a unique normal conditional expectation E from $\mathcal{N}$ onto $J(\mathcal{M})$ such that

$$\varphi \circ E = \varphi.$$

From this, we may induce a completely positive and completely contractive projection E_p from $L_p(\mathcal{N}) = L_p(\mathcal{N}, \varphi)$ onto $w^*T(L_p(\mathcal{M}, \tau))$ (see [11], [16], [38], or the construction sketched after Lemma 4.8). Then $h \mapsto wE_p(w^*h)$ is the required projection. If T is positive, w and w^* may be omitted, so this projection is E_p itself. □

Proposition 3.3 is a noncommutative version of the classical fact that for any isometric embedding

$$T\colon L_p(X, \Sigma_X, \mu_X) \to L_p(Y, \Sigma_Y, \mu_Y),$$

the image space $T(L_p(X, \Sigma_X, \mu_X))$ must be contractively complemented in the space $L_p(Y, \Sigma_Y, \mu_Y)$ (see Lacey [23]). Stronger results are in Theorem 4.9 of this paper and Section 8 of [38].

4. 2-isometries on general noncommutative L_p-spaces

The aim of this section is to study 2-isometries on general noncommutative L_p-spaces. We will use the alternative notation, i.e. positive linear functionals $\varphi^{\frac{1}{p}}$, instead of positive self-adjoint operators $h_\varphi^{\frac{1}{p}}$, for elements in $L_p(\mathcal{M})^+$. Until the proof of Theorem 2 given at the end of the section, we assume that $\mathcal{M}$ is an arbitrary σ-finite von Neumann algebra with a fixed normal faithful state ϕ. We also use ϕ for its corresponding density operator h_ϕ in $L_1(\mathcal{M})$, so that $\phi^{\frac{1}{p}}\mathcal{M}$ is norm dense in $L_p(\mathcal{M})$.

As in §3, we start by using support projections to construct an embedding from $\mathcal{M}$ into $\mathcal{N}$. But since $\mathcal{M} \bigcap L_p(\mathcal{M}) = \{0\}$ ($p \neq \infty$) when $\mathcal{M}$ is not semifinite, we cannot directly employ the projection lattice of $\mathcal{M}$, and instead work with the vectors $\{\varphi^{\frac{1}{p}} x \mid x \in \mathcal{M}\}$. Then 2×2 matrix equations produce the *-monomorphism π (in the next proposition), but several steps are still required to show that $\pi(\mathcal{M})$ is complemented and obtain the desired decomposition for T.

Proposition 4.1. *Let $1 \leq p < \infty$ and $p \neq 2$. If $T\colon L_p(\mathcal{M}) \to L_p(\mathcal{N})$ is a 2-isometry, then there exists a normal *-monomorphism $\pi\colon \mathcal{M} \to \mathcal{N}$ such that*

$$T(\phi^{\frac{1}{p}} x) \;=\; T(\phi^{\frac{1}{p}})\pi(x)\;, \qquad x \in \mathcal{M}. \tag{4.1}$$

Moreover, π does not depend on the choice of (faithful) $\phi \in \mathcal{M}_*^+$.

Proof. We first define a map π between projection lattices by

$$\pi\colon e \in \mathcal{P}(\mathcal{M}) \mapsto s_r(T(\phi^{\frac{1}{p}} e)) \in \mathcal{P}(\mathcal{N}). \tag{4.2}$$

That is, with the polar decomposition $T(\phi^{\frac{1}{p}} e) = w_e |T(\phi^{\frac{1}{p}} e)|$, we set $\pi(e) = w_e^* w_e$. If $e \perp f$, then

$$\begin{bmatrix} \phi^{\frac{1}{p}} e & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ \phi^{\frac{1}{p}} f & 0 \end{bmatrix}$$

are two orthogonal elements in $S_p^2[L_p(\mathcal{M})]$. By Theorem 2.2, their images

$$\begin{bmatrix} T(\phi^{\frac{1}{p}}e) & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ T(\phi^{\frac{1}{p}}f) & 0 \end{bmatrix}$$

under $\mathrm{id}_{S_p^2} \otimes T$ are orthogonal in $S_p^2[L_p(\mathcal{N})]$. This shows that

$$s_r\left(\begin{bmatrix} T(\phi^{\frac{1}{p}}e) & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} s_r(T(\phi^{\frac{1}{p}}e)) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \pi(e) & 0 \\ 0 & 0 \end{bmatrix}$$

is orthogonal to

$$s_r\left(\begin{bmatrix} 0 & 0 \\ T(\phi^{\frac{1}{p}}f) & 0 \end{bmatrix}\right) = \begin{bmatrix} s_r(T(\phi^{\frac{1}{p}}f)) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \pi(f) & 0 \\ 0 & 0 \end{bmatrix},$$

and thus $\pi(e)$ is orthogonal to $\pi(f)$. Since

$$\begin{aligned} T(\phi^{\frac{1}{p}})\pi(e) &= [T(\phi^{\frac{1}{p}}e) + T(\phi^{\frac{1}{p}}(1-e))]\pi(e) \\ &= T(\phi^{\frac{1}{p}}e) + T(\phi^{\frac{1}{p}})\pi(1-e)\pi(e) = T(\phi^{\frac{1}{p}}e), \end{aligned}$$

we have (4.1) for projections in $\mathcal{M}$. As a consequence, we obtain

$$\pi(e+f) = \pi(e) + \pi(f)$$

for orthogonal projections $e \perp f$ since

$$T(\phi^{\frac{1}{p}})\pi(e+f) = T(\phi^{\frac{1}{p}}(e+f)) = T(\phi^{\frac{1}{p}}e) + T(\phi^{\frac{1}{p}}f) = T(\phi^{\frac{1}{p}})(\pi(e) + \pi(f))$$

and $T(\phi^{\frac{1}{p}})$ is separating for the right action of $\pi(1)\mathcal{N}\pi(1)$.

Now we extend π as in the proof of Theorem 3.1: first to finite real linear combinations of orthogonal projections, then to all self-adjoint elements in $\mathcal{M}$ by continuity, and finally to all of $\mathcal{M}$ by complex linearity. Apparently π satisfies

$$T(\phi^{\frac{1}{p}}x) = T(\phi^{\frac{1}{p}})\pi(x), \qquad x \in \mathcal{M}.$$

This relation implies additivity: for $x, y \in \mathcal{M}$, we have

$$T(\phi^{\frac{1}{p}})\pi(x+y) = T(\phi^{\frac{1}{p}}(x+y)) = T(\phi^{\frac{1}{p}}x) + T(\phi^{\frac{1}{p}}y) = T(\phi^{\frac{1}{p}})(\pi(x) + \pi(y)).$$

Now let u be a unitary element of $\mathcal{M}$. Replacing $\phi^{\frac{1}{p}}$ by $\phi^{\frac{1}{p}}u$ in the definition of π, we obtain a new *-preserving map π_u. For a projection e in $\mathcal{M}$,

$$\begin{bmatrix} \phi^{\frac{1}{p}}e & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ \phi^{\frac{1}{p}}u(1-e) & 0 \end{bmatrix}$$

are orthogonal, so $\pi(e) \perp \pi_u(1-e)$. Since

$$\pi_u(1) = s_r(T(\phi^{\frac{1}{p}}u)) = s_r(\pi(u)) = \pi(1),$$

we must have $\pi(e) = \pi_u(e)$ for every projection e, whence $\pi = \pi_u$. Then for any $x \in \mathcal{M}$,

$$T(\phi^{\frac{1}{p}})\pi(ux) = T(\phi^{\frac{1}{p}}ux) = T(\phi^{\frac{1}{p}}u)\pi(x) = T(\phi^{\frac{1}{p}})\pi(u)\pi(x).$$

Since the linear span of the unitaries is all of $\mathcal{M}$, it follows that π is multiplicative. Then

$$T(\phi^{\frac{1}{p}}xy) = T(\phi^{\frac{1}{p}})\pi(xy) = T(\phi^{\frac{1}{p}})\pi(x)\pi(y) = T(\phi^{\frac{1}{p}}x)\pi(y), \qquad x, y \in \mathcal{M},$$

(densely) establishes (4.1) and the independence of π from the choice of ϕ.

To prove the normality of π, we let $\{x_\alpha\}$ be a bounded net in $\mathcal{M}$ such that $x_\alpha \xrightarrow{s} x$ in the strong topology. It follows from Lemma 2.1 that $\phi^{\frac{1}{p}}x_\alpha \to \phi^{\frac{1}{p}}x$ in $L_p(\mathcal{M})$, and thus

$$T(\phi^{\frac{1}{p}})\pi(x_\alpha) = T(\phi^{\frac{1}{p}}x_\alpha) \to T(\phi^{\frac{1}{p}}x) = T(\phi^{\frac{1}{p}})\pi(x)$$

in $L_p(\mathcal{N})$. Again by Lemma 2.1, this implies $\pi(x_\alpha) \xrightarrow{s} \pi(x)$ in $\pi(1)\mathcal{N}\pi(1)$ and thus in $\mathcal{N}$. □

Duality will be a very important tool in the following arguments. Given a bounded linear map T: $L_p(\mathcal{M}) \to L_p(\mathcal{N})$, we let T': $L_{p'}(\mathcal{N}) \to L_{p'}(\mathcal{M})$ denote the adjoint of T and let $T^{'*} = (T')^*$ denote the *-adjoint of T', i.e.,

$$T^{'*}(k) = T'(k^*)^* \qquad k \in L_p(\mathcal{N}).$$

Then

$$tr_{\mathcal{M}}(T^{'*}(k)^*h) \;=\; tr_{\mathcal{N}}(k^*T(h))\;, \qquad h \in L_p(\mathcal{M}),\, k \in L_{p'}(\mathcal{N}),$$

defines a sesquilinear form on $L_p(\mathcal{M}) \times L_{p'}(\mathcal{N})$. We also set the notation $\bar{\varphi} \triangleq |T(\varphi^{\frac{1}{p}})|^p$ for any state $\varphi \in \mathcal{M}_*^+$. Thus $\bar{\varphi}^{\frac{1}{p}}$ is the absolute value of $T(\varphi^{\frac{1}{p}})$.

Lemma 4.2. *Let* $1 \le p < \infty$ *and let* T: $L_p(\mathcal{M}) \to L_p(\mathcal{N})$ *be a* 2*-isometry (or an isometry satisfying* (4.1)*). Then*

$$\bar{\phi} \circ \pi \;=\; \phi\;.$$

Proof. Let us first assume that $p \neq 1$. Since ϕ is a normal faithful state in $\mathcal{M}_*^+$, then $\phi^{\frac{1}{p}}$ is a unit vector in $L_p(\mathcal{M})$ since

$$\|\phi^{\frac{1}{p}}\|_p^p = \phi(1) = 1.$$

This implies that $T(\phi^{\frac{1}{p}}) = w\bar{\phi}^{\frac{1}{p}}$ is a unit vector in $L_p(\mathcal{N})$ and thus

$$1 \;=\; tr_{\mathcal{N}}((w\bar{\phi}^{\frac{1}{p'}})^*T(\phi^{\frac{1}{p}})) \;=\; tr_{\mathcal{M}}(T^{'*}(w\bar{\phi}^{\frac{1}{p'}})^*\phi^{\frac{1}{p}})\;.$$

Since $L_{p'}(\mathcal{M})$ is uniformly convex, $\phi^{\frac{1}{p}}$ admits exactly one norm attaining element. Thus we must have

$$T^{'*}(w\bar{\phi}^{\frac{1}{p'}}) \;=\; \phi^{\frac{1}{p'}}\;.$$

On the other hand, we have

$$\begin{aligned}\phi(x) &= tr_{\mathcal{M}}((\phi^{\frac{1}{p'}})^*\phi^{\frac{1}{p}}x) \;=\; tr_{\mathcal{M}}(T^{'*}(w\bar{\phi}^{\frac{1}{p'}})^*\phi^{\frac{1}{p}}x) \;=\; tr_{\mathcal{N}}((w\bar{\phi}^{\frac{1}{p'}})^*T(\phi^{\frac{1}{p}}x))\\ &= tr_{\mathcal{N}}(\bar{\phi}^{\frac{1}{p'}}w^*w\bar{\phi}^{\frac{1}{p}}\pi(x)) \;=\; \bar{\phi}(\pi(x))\;.\end{aligned}$$

If $p = 1$, the same argument applies, replacing $\bar{\phi}^{\frac{1}{p'}}$ and $\phi^{\frac{1}{p'}}$ by 1. (Since ϕ is faithful, it attains its norm at 1 only.) $\square$

We can already say quite a lot in case $p = 1$.

Proposition 4.3. *Let T: $\mathcal{M}_* \to \mathcal{N}_*$ be a 2-isometry. There are a normal *-monomorphism π: $\mathcal{M} \to \mathcal{N}$, a partial isometry $w \in \mathcal{N}$, and a normal conditional expectation E: $\mathcal{N} \to \pi(\mathcal{M})$ such that*

$$T(\varphi) = w(\varphi \circ \pi^{-1} \circ E), \qquad \varphi \in \mathcal{M}_*.$$

Proof. It was shown in [38, Theorem 3.2] that for any such L_1-isometry T, there are a normal Jordan *-monomorphism J: $\mathcal{M} \to \mathcal{N}$, a partial isometry $w \in \mathcal{N}$, and a normal positive projection P: $\mathcal{N} \to J(\mathcal{M})$, faithful on $J(1)\mathcal{N}J(1)$, such that

$$T(\varphi) = w(\varphi \circ J^{-1} \circ P), \qquad \varphi \in \mathcal{M}_*.$$

(We note that an equivalent result was proved earlier by Kirchberg [20].) So the induced map $S = w^*T$ is a positive isometry and thus we can get $\bar{\varphi} = \varphi \circ J^{-1} \circ P$ for all $\varphi \in \mathcal{M}_*^+$. Precomposing with the π from Proposition 4.1 and applying Lemma 4.2,

$$\varphi \circ J^{-1} \circ P \circ \pi = \bar{\varphi} \circ \pi = \varphi, \qquad \forall \text{ faithful } \varphi \in \mathcal{M}_*^+.$$

Apparently $J^{-1} \circ P \circ \pi$ is the identity map. In particular, we have $P(\pi(e)) = J(e)$ for any projection e in $\mathcal{M}$. Using properties of P and J (see [38, Lemma 5.4]), we can conclude that

$$P(J(e)\pi(e)J(e)) = J(e)P(\pi(e))J(e) = [J(e)]^3 = J(e) = P(J(e)).$$

Since $J(e)\pi(e)J(e) \le J(e)$ and $J(1) = s(P)$, we must have $J(e)\pi(e)J(e) = J(e)$. This implies $\pi(e) \ge J(e)$. Since $P(\pi(e) - J(e)) = 0$, we must have $\pi(e) = J(e)$. Therefore $J = \pi$ is multiplicative and $E = P$ is a normal conditional expectation from $\mathcal{N}$ onto $\pi(\mathcal{M})$. $\square$

Lemma 4.4. *Let $\mathcal{N}_1$ be a von Neumann subalgebra of $\mathcal{N}_2$ and let $\bar{\phi}$ be a normal faithful state on $\mathcal{N}_2$ with $\phi = \bar{\phi}|_{\mathcal{N}_1}$. For $2 \le p \le \infty$,*

$$\left\|\bar{\phi}^{\frac{1}{p}} x\right\|_{L_p(\mathcal{N}_2)} \le \left\|\phi^{\frac{1}{p}} x\right\|_{L_p(\mathcal{N}_1)}$$

for all $x \in \mathcal{N}_1$.

Proof. We first recall from Section 2 that we can identify Haagerup's L_p-space $L_p(\mathcal{N}_i)$ with the complex interpolation spaces $[\mathcal{N}_i, (\mathcal{N}_i)_*]_{\frac{1}{p}}$ $(i = 1, 2)$ (see (2.5) and (2.6)). Since $\phi = \bar{\phi}|_{\mathcal{N}_1}$ and $\bar{\phi}$ are normal faithful states on $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, the induced inclusion

$$\iota_2\colon \phi^{\frac{1}{2}} x \in L_2(\mathcal{N}_1, \phi) \to \bar{\phi}^{\frac{1}{2}} x \in L_2(\mathcal{N}_2, \bar{\phi})$$

is an isometric inclusion. The corresponding inclusion

$$\iota_2\colon \phi x \in [\mathcal{N}_1, (\mathcal{N}_1)_*]_{\frac{1}{2}} \to \bar{\phi} x \in [\mathcal{N}_2, (\mathcal{N}_2)_*]_{\frac{1}{2}}$$

between complex interpolation spaces is also an isometric inclusion. Then for any $2 \le p < \infty$ the canonical inclusion

$$\iota_p\colon \phi^{\frac{1}{p}} x \in L_p(\mathcal{N}_1) \to \bar{\phi}^{\frac{1}{p}} x \in L_p(\mathcal{N}_2)$$

is a contraction since it can be identified with the complex interpolation

$$\iota_p = [\iota_\infty, \iota_2]_{\frac{2}{p}},$$

where we let $\iota_\infty\colon \mathcal{N}_1 \hookrightarrow \mathcal{N}_2$ denote the canonical inclusion of $\mathcal{N}_1$ into $\mathcal{N}_2$. □

Getting back to T, let us assume that

$$T(\phi^{\frac{1}{p}} x) = \bar{\phi}^{\frac{1}{p}} \pi(x).$$

Otherwise, we may replace T with $\phi^{\frac{1}{p}} x \mapsto w^* T(\phi^{\frac{1}{p}} x)$ where w is the partial isometry obtain from the polar decomposition $T(\phi^{\frac{1}{p}}) = w\bar{\phi}^{\frac{1}{p}}$. For any $1 \le q < \infty$, we define the related maps

$$T_q(\phi^{\frac{1}{q}} x) = \bar{\phi}^{\frac{1}{q}} \pi(x) .$$

Corollary 4.5. *Let $1 < p \le 2$. If $T = T_p\colon \phi^{\frac{1}{p}} x \mapsto \bar{\phi}^{\frac{1}{p}} \pi(x)$ is an isometry, then $T_{p'}$ is also an isometry.*

Proof. Let us show that $T_{p'}^{'*}T_p$ is the identity on $L_p(\mathcal{M})$. Indeed, by definition and Lemma 4.2 we find

$$tr_{\mathcal{M}}(T_{p'}^{'*}(\bar{\phi}^{\frac{1}{p}}\pi(y))^*\phi^{\frac{1}{p'}}x) = tr_{\mathcal{M}}((\bar{\phi}^{\frac{1}{p}}\pi(y))^*T_{p'}(\phi^{\frac{1}{p'}}x)) = tr_{\mathcal{N}}(\pi(y^*)\bar{\phi}^{\frac{1}{p}}\bar{\phi}^{\frac{1}{p'}}\pi(x))$$
$$= \bar{\phi}(\pi(xy^*)) \;=\; \phi(xy^*) = tr_{\mathcal{M}}((\phi^{\frac{1}{p}}y)^*\phi^{\frac{1}{p'}}x)\;.$$

By duality we conclude that $T_{p'}^{'*}(\bar{\phi}^{\frac{1}{p}}\pi(y)) = \phi^{\frac{1}{p}}y$. This implies that

$$T_{p'}^{'*}(T_p(\phi^{\frac{1}{p}}y))) = T_{p'}^{'*}(\bar{\phi}^{\frac{1}{p}}\pi(x)) = \phi^{\frac{1}{p}}y.$$

This shows that $T_{p'}^{'*}T_p = \mathrm{id}_{L_p(\mathcal{M})}$. By duality, we deduce that

$$\mathrm{id}_{L_{p'}(\mathcal{M})} \;=\; (T_{p'}^{'*}T_p)^{'*} \;=\; T_p^{'*}T_{p'}\;.$$

By assumption T_p and thus $T_p^{'*}$ is a contraction. $T_{p'}$ is also contractive by Lemma 4.4, so it must be an isometry. □

Due to Corollary 4.5, we may now focus on the case $2 < p < \infty$. The main argument here is to show that if T_p is isometric for some value of p, then it is isometric for all values. We will use Kosaki's interpolation theorem for the proof of this result but in a more explicit form. Let us use the notation $\mathfrak{S} = \{z \in \mathbb{C}|\; 0 \le Re(z) \le 1\}$.

Lemma 4.6. *Let ϕ and ψ be normal faithful states such that $\psi \le C\phi$ for some constant $C > 0$. Let $2 < r < p < q < \infty$ and $\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{r}$. Then there exists an analytic function h: $\mathfrak{S} \to \mathcal{M}$ such that*

(1) $\left\|\phi^{\frac{1}{q}}h(it)\right\|_q \le 1$,
(2) $\left\|\phi^{\frac{1}{r}}h(1+it)\right\|_r \le 1$,
(3) $\phi^{\frac{1}{q}}h(0) = \psi^{\frac{1}{q}}$, $\phi^{\frac{1}{p}}h(\theta) \;=\; \psi^{\frac{1}{p}}$, $\phi^{\frac{1}{r}}h(1) = \psi^{\frac{1}{r}}$,
(4) *the maps* $t \mapsto h(1+it)\phi^{\frac{1}{r}}$, $t \mapsto h(it)\phi^{\frac{1}{q}}$ *are continuous.*

Proof. Since $\psi \le C\phi$, we have by [44, Theorem VIII.3.17] (taking adjoints) that the Connes cocycle derivative $(D\phi : D\psi)_t = \psi^{it}\phi^{-it}$ extends off the real line to a σ-weakly continuous function on $\{z \mid 0 \le \mathrm{Im}\, z \le 1/2\}$ which is analytic in the interior. We define $\frac{1}{s} \;=\; \frac{1}{r} - \frac{1}{q}$ and set

$$h(z) = (D\phi : D\psi)_{i\left(\frac{1}{q}+\frac{z}{s}\right)} = \phi^{-\frac{1}{q}-\frac{z}{s}}\psi^{\frac{1}{q}+\frac{z}{s}}.$$

Note that h is analytic on $\mathfrak{S}$.

We have

$$\left\|\phi^{\frac{1}{q}}h(it)\right\| = \left\|(\phi^{-\frac{it}{s}}\psi^{\frac{it}{s}})\psi^{\frac{1}{q}}\right\|_q \;=\; \left\|\psi^{\frac{1}{q}}\right\|_q \;=\; 1.$$

Similarly,

$$\left\|\phi^{\frac{1}{r}}h(1+it)\right\|_r \;=\; \left\|(\phi^{-\frac{it}{s}}\psi^{\frac{it}{s}})\psi^{\frac{1}{r}}\right\|_q \;=\; 1\;.$$

The equalities for $\phi^{\frac{1}{q}}h(0)$, $\phi^{\frac{1}{r}}h(1)$ and $\phi^{\frac{1}{p}}h(\theta)$ are obvious. Since $(\phi^{-\frac{it}{s}}\psi^{\frac{it}{s}})$ is a cocycle, it is strongly continuous. Then we obtain the last statement by Lemma 2.1 and the above equations. □

We are now able to prove the key extrapolation result.

Proposition 4.7. *Let ϕ be a normal faithful state on $\mathcal{N}$, $\bar{\phi}\circ\pi = \phi$, and $T_p(\phi^{\frac{1}{p}}x) = \bar{\phi}^{\frac{1}{p}}\pi(x)$. If T_p is an isometry for some $2<p<\infty$, then T_p is an isometry for all $2<p<\infty$.*

Proof. Let $2<r<p<q<\infty$ and assume that T_p is an isometry. We want to show that T_q and T_r are isometries. We define $\frac{1}{s}=\frac{1}{r}-\frac{1}{q}$ and θ by $\frac{1}{p}=\frac{1-\theta}{q}+\frac{\theta}{r}$. Let us assume that ψ is a normal faithful state with $\psi\le C\phi$ for some constant $C<\infty$. (Such ψ form a dense face of $\mathcal{M}_*^+$, see e.g. [13].) We know that T_q is a contraction by Lemma 4.4; assume toward a contradiction that

$$\left\|T_q(\psi^{\frac{1}{q}})\right\|_q < 1\;.$$

Since T_q is continuous, we can find a $\delta>0$ such that

$$\left\|T_q(\phi^{-\frac{it}{s}}\psi^{\frac{it}{s}}\psi^{\frac{1}{q}})\right\|_q \le (1-\delta) \tag{4.3}$$

for all $|t|\le\delta$. Let μ_θ be the probability measure on the boundary of the strip $\mathfrak{S}$ such that

$$\int_{\partial\mathfrak{S}} f(z)d\mu_\theta(z) \;=\; f(\theta)$$

for every harmonic function on $\mathfrak{S}$. This is just a relocation of the Poisson kernel, so Lebesgue measure is absolute continuous with respect to μ_θ (an explicit formula is in [3, Section 4.3, p.93]). Therefore $\mu_\theta([-i\delta,i\delta])>0$. So we may also find $\varepsilon>0$ such that $\frac{1-\varepsilon}{1+\varepsilon}>1-\delta\mu_\theta[-i\delta,i\delta]$. Since $L_{p'}(\mathcal{N}) \;=\; [L_{q'}(\mathcal{N}),L_{r'}(\mathcal{N})]_\theta$, we may find an approximately norm-attaining element for $T_p(\psi^{\frac{1}{p}})$ as a simple element of the interpolation space. That is, we find an analytic function $g:\mathfrak{S}\to L_{r'}(\mathcal{N})$, continuous on the boundary and vanishing at ∞ such that

$$\left\|g(it)\bar{\phi}^{\frac{1}{r}-\frac{1}{q}}\right\|_{q'} \le (1+\varepsilon), \qquad \|g(1+it)\|_{r'} \;\le\; (1+\varepsilon),$$

and

$$1-\varepsilon \le |tr_{\mathcal{N}}(g(\theta)\bar{\phi}^{\frac{1}{r}-\frac{1}{p}}T_p(\psi^{\frac{1}{p}}))| \ .$$

Let h be the function given by Lemma 4.6, and set

$$F(z) \ = \ tr_{\mathcal{N}}(g(z)\bar{\phi}^{\frac{1}{r}}\pi(h(z))) \ = \ tr_{\mathcal{N}}(g(z)T_r(\phi^{\frac{1}{r}}h(z))) \ .$$

Since $z \mapsto \phi^{\frac{1}{r}}h(z)$ is analytic in the strip and continuous at the boundary and bounded, we know that F is analytic, continuous at the boundary and vanishes at ∞. Therefore, we deduce that

$$\begin{aligned}(4.4)\quad 1-\varepsilon &\le |tr(g(\theta)\bar{\phi}^{\frac{1}{r}-\frac{1}{p}}T_p(\phi^{\frac{1}{p}}h(\theta))| = |tr(g(\theta)\bar{\phi}^{\frac{1}{r}}\pi(h(\theta))| = |F(\theta)| \\ &= \left|\int F(z)d\mu_\theta(z)\right| \ \le \ \int_{i\mathbb{R}} |F(it)|d\mu_\theta(it) + \int_{1+i\mathbb{R}} |F(1+it)|d\mu_\theta(it) \ .\end{aligned}$$

For all t we have

$$\begin{aligned}|F(it)| &= |tr_{\mathcal{N}}(g(it)T_r(\phi^{\frac{1}{r}}h(it))| \ = \ |tr_{\mathcal{N}}(g(it)\bar{\phi}^{\frac{1}{r}}\pi(h(it)))| \\ &= |tr_{\mathcal{N}}(g(it)\bar{\phi}^{\frac{1}{r}-\frac{1}{q}}T_q(\phi^{\frac{1}{q}}h(it))| \\ &\le \left\|g(it)\bar{\phi}^{\frac{1}{r}-\frac{1}{q}}\right\|_{q'} \left\|\phi^{\frac{1}{q}}h(it)\right\|_q \ \le \ (1+\varepsilon) \ .\end{aligned}$$

However, for $|t| \le \delta$, we note by (4.3) that the inequality is stronger:

$$\begin{aligned}|F(it)| &= |tr_{\mathcal{N}}(g(it)T_r(\phi^{\frac{1}{r}}h(it))| \ = \ |tr_{\mathcal{N}}(g(it)\bar{\phi}^{\frac{1}{r}}\pi(h(it)))| \\ &= |tr_{\mathcal{N}}(g(it)\bar{\phi}^{\frac{1}{r}-\frac{1}{q}}T_q(\phi^{\frac{1}{q}}h(it))| \\ &\le \left\|g(it)\bar{\phi}^{\frac{1}{r}-\frac{1}{q}}\right\|_{q'} \left\|T(\phi^{\frac{1}{q}}h(it))\right\|_q \\ &\le (1+\varepsilon)\left\|T(\phi^{-\frac{it}{s}}\psi^{\frac{it}{s}}\psi^{\frac{1}{q}})\right\|_q \\ &\le (1+\varepsilon)(1-\delta) \ .\end{aligned}$$

Similarly, we find

$$|F(1+it)| \ \le \ 1+\varepsilon \ .$$

By (4.4), this implies

$$\begin{aligned}1-\varepsilon &\le (1+\varepsilon)[(1-\delta)\mu_\theta([-i\delta,i\delta]) + \mu_\theta(\partial\mathfrak{S}\setminus[-i\delta,i\delta])] \\ &= (1+\varepsilon)[1-\delta\mu_\theta([-i\delta,i\delta])] \ ,\end{aligned}$$

contradicting the choice of ε. This shows that $\left\|T_q(\psi^{\frac{1}{q}})\right\|_q = 1$. The argument for $\left\|T_r(\psi^{\frac{1}{r}})\right\|_q = 1$ is similar.

It was shown in [33] that the Mazur map $\psi \mapsto \psi^{\frac{1}{q}}$ is continuous. So the above argument applies to a dense set of $L_q(M)_+$. Since T_q is a contraction, we deduce for all positive elements $\psi^{\frac{1}{q}}$

$$\left\| T_q(\psi^{\frac{1}{q}}) \right\|_{L_q(\mathcal{N})} \;=\; \|\psi^{\frac{1}{q}}\|_{L_q(\mathcal{M})} \;.$$

For an arbitrary element $h \in L_q(\mathcal{M})$, we write $hv^* = |\xi^*|$ where v is a partial isometry. Then

$$\|h\|_q = \|h^*\|_q \;=\; \|\,|h^*|\,\|_q \;=\; \|T_q(|h^*|)\|_q \;=\; \|T_q(h)\pi(v^*)\|_q \leq \|T_q(h)\|_q \;.$$

Therefore T_q is an isometry. The same argument applies for T_r. □

Lemma 4.8. *Keep the notations of the previous proposition, and let T_4 be a 2-isometry. Then for all $x \in \mathcal{M}$,*

$$\left\| \bar{\phi}^{\frac{1}{4}} \pi(x) \bar{\phi}^{\frac{1}{4}} \right\|_2 \;=\; \left\| \phi^{\frac{1}{4}} x \phi^{\frac{1}{4}} \right\|_2 \;.$$

Proof. Let x be a positive element, then we know that

$$\left\| \bar{\phi}^{\frac{1}{4}} \pi(x) \bar{\phi}^{\frac{1}{4}} \right\|_2 \;=\; \left\| \bar{\phi}^{\frac{1}{4}} \pi(x^{\frac{1}{2}}) \right\|_4^2 \;=\; \left\| \phi^{\frac{1}{4}} x^{\frac{1}{2}} \right\|_4^2 \;=\; \left\| \phi^{\frac{1}{4}} x \phi^{\frac{1}{4}} \right\|_2 \;.$$

Given an arbitrary element $x \in \mathcal{M}$ of norm less than one, the matrix

$$\tilde{x} \;=\; \begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix}$$

is positive. We apply the observation above to $\tilde{x}$ and $\mathrm{tr}_2 \otimes \phi$ as a normal faithful state on $M_2(\mathcal{M})$ and deduce

$$\left\| (\mathrm{tr}_2 \otimes \bar{\phi})^{\frac{1}{4}} \pi_2(\tilde{x}) (\mathrm{tr}_2 \otimes \bar{\phi})^{\frac{1}{4}} \right\|_2 \;=\; \left\| (\mathrm{tr}_2 \otimes \phi)^{\frac{1}{4}} \tilde{x} (\mathrm{tr}_2 \otimes \phi)^{\frac{1}{4}} \right\|_2 \;.$$

Since $\left\| \phi^{\frac{1}{4}} x^* \phi^{\frac{1}{4}} \right\|_2^2 = \left\| \phi^{\frac{1}{4}} x \phi^{\frac{1}{4}} \right\|_2^2$, we have

$$\begin{aligned}
&\left\| (\mathrm{tr}_2 \otimes \phi)^{\frac{1}{4}} \tilde{x} (\mathrm{tr}_2 \otimes \phi)^{\frac{1}{4}} \right\|_2^2 \\
&= (\mathrm{tr}_2 \otimes tr_{\mathcal{M}}) \left(\begin{bmatrix} \phi^{\frac{1}{4}} & 0 \\ 0 & \phi^{\frac{1}{4}} \end{bmatrix} \begin{bmatrix} 1 & x^* \\ x & 1 \end{bmatrix} \begin{bmatrix} \phi^{\frac{1}{2}} & 0 \\ 0 & \phi^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix} \begin{bmatrix} \phi^{\frac{1}{4}} & 0 \\ 0 & \phi^{\frac{1}{4}} \end{bmatrix} \right) \\
&= (\mathrm{tr}_2 \otimes tr_{\mathcal{M}}) \left(\begin{bmatrix} \phi + \phi^{\frac{1}{4}} x^* \phi^{\frac{1}{2}} x \phi^{\frac{1}{4}} & \phi^{\frac{3}{4}} x \phi^{\frac{1}{4}} + \phi^{\frac{1}{4}} x^* \phi^{\frac{3}{4}} \\ \phi^{\frac{3}{4}} x \phi^{\frac{1}{4}} + \phi^{\frac{1}{4}} x^* \phi^{\frac{3}{4}} & \phi + \phi^{\frac{1}{4}} x \phi^{\frac{1}{2}} x^* \phi^{\frac{1}{4}} \end{bmatrix} \right) \\
&= \frac{1}{2} \left(2tr_{\mathcal{M}}(\phi) + tr_{\mathcal{M}}(\phi^{\frac{1}{4}} x^* \phi^{\frac{1}{2}} x \phi^{\frac{1}{4}}) + tr_{\mathcal{M}}(\phi^{\frac{1}{4}} x \phi^{\frac{1}{2}} x^* \phi^{\frac{1}{4}}) \right) \\
&= 1 + \left\| \phi^{\frac{1}{4}} x \phi^{\frac{1}{4}} \right\|_2^2 .
\end{aligned}$$

Since $\bar{\phi}(\pi(1)) = 1$, the same calculation shows that

$$\left\|(\mathrm{tr}_2 \otimes \bar{\phi})^{\frac{1}{4}} \pi_2(\tilde{x}) (\mathrm{tr}_2 \otimes \bar{\phi})^{\frac{1}{4}}\right\|_2^2 = \left(1 + \left\|\bar{\phi}^{\frac{1}{4}} \pi(x) \bar{\phi}^{\frac{1}{4}}\right\|_2^2\right).$$

This proves the assertion. □

At this point we recall how complete isometries of L_p-spaces can be constructed from either *-isomorphisms or conditional expectations. Details for the constructions of the next two paragraphs can be found in [11], [16], and [38].

Let π: $\mathcal{M}_1 \to \mathcal{M}_2$ be a surjective *-isomorphism, and let $\phi \in (\mathcal{M}_1)_*^+$ be faithful. Then there is an associated completely isometric isomorphism π_p: $L_p(\mathcal{M}_1) \xrightarrow{\sim} L_p(\mathcal{M}_2)$, densely defined by

$$\phi^{\frac{1}{p}} x \mapsto (\phi \circ \pi^{-1})^{\frac{1}{p}} \pi(x), \qquad x \in \mathcal{M}_1.$$

Now consider the situation where $\mathcal{M}_1$ is a conditioned σ-finite subalgebra of $\mathcal{M}_2$, so that there are a normal *-isomorphism ι: $\mathcal{M}_1 \hookrightarrow \mathcal{M}_2$ (thought of as the identity map) and a normal conditional expectation E: $\mathcal{M}_2 \twoheadrightarrow \mathcal{M}_1$. By interpolation one can find a complete isometry ι_p: $L_p(\mathcal{M}_1) \hookrightarrow L_p(\mathcal{M}_2)$ and a complete contraction E_p: $L_p(\mathcal{M}_2) \twoheadrightarrow \iota_p(L_p(\mathcal{M}_1))$. Both ι_p and E_p are completely positive. With $\phi \in (\mathcal{M}_1)_*^+$ faithful, ι_p is densely defined by

$$\phi^{\frac{1}{p}} x \mapsto (\phi \circ E)^{\frac{1}{p}} x, \qquad x \in \mathcal{M}_1.$$

Theorem 4.9. *Let $1 \leq p \neq 2 < \infty$. A linear map T: $L_p(\mathcal{M}) \to L_p(\mathcal{N})$ is a 2-isometry if and only if there exist a normal *-monomorphism π: $\mathcal{M} \to \mathcal{N}$, a partial isometry $w \in \mathcal{N}$, and a normal conditional expectation E: $\mathcal{N} \to \pi(\mathcal{M})$ such that $\pi(1) = w^*w$ and*

$$T(\phi^{\frac{1}{p}} x) = w(\phi \circ \pi^{-1} \circ E)^{\frac{1}{p}} \pi(x), \qquad \forall x \in \mathcal{M}. \tag{4.5}$$

Under these conditions, the range $T(L_p(\mathcal{M}))$ is completely contractively complemented, and T is a module map:

$$T(hx) = T(h)\pi(x), \qquad \forall x \in \mathcal{M},\ h \in L_p(\mathcal{M}).$$

Proof. First note that maps of the form (4.5) are always complete isometries with completely contractively complemented range: they decompose as π_p, then ι_p, then left multiplication by w. Each of these is a complete isometry, and the image is the range of the complete contraction $L_p(\mathcal{M}_2) \ni h \mapsto wE_p(w^*h)$.

Now we turn to the derivation of (4.5) for 2-isometries. The case $p = 1$ has been proved in Proposition 4.3. To prove the result for general p, we may first apply Proposition 4.1 and assume that

$$T(\phi^{\frac{1}{p}}x) \ = \ \bar{\phi}^{\frac{1}{p}}\pi(x) \ .$$

We can obtain the result for general T by multiplying by an appropriate partial isometry.

According to Lemma 4.2, we also know that $\phi \ = \ \bar{\phi} \circ \pi$. If T is a 2-isometry for some $1 < p \neq 2 < \infty$, then we can claim from Corollary 4.5 and Proposition 4.7 that T_4 is a 2-isometry. Then Lemma 4.8 shows that

$$\left\|\phi^{\frac{1}{4}}x\phi^{\frac{1}{4}}\right\|_2 \ = \ \left\|\bar{\phi}^{\frac{1}{4}}\pi(x)\bar{\phi}^{\frac{1}{4}}\right\|_2 \ .$$

By polarization, we deduce

$$(\phi^{\frac{1}{4}}x\phi^{\frac{1}{4}}, \phi^{\frac{1}{4}}y\phi^{\frac{1}{4}}) \ = \ (\bar{\phi}^{\frac{1}{4}}\pi(x)\bar{\phi}^{\frac{1}{4}}, \bar{\phi}^{\frac{1}{4}}\pi(y)\bar{\phi}^{\frac{1}{4}})$$

for all $x, y \in \mathcal{M}$. This means that

$$tr_{\mathcal{M}}(\phi^{\frac{1}{2}}x\phi^{\frac{1}{2}}y^*) \ = \ tr_{\mathcal{N}}(\bar{\phi}^{\frac{1}{2}}\pi(x)\bar{\phi}^{\frac{1}{2}}\pi(y)^*) \ .$$

In other words the sesquilinear selfpolar form [51] $s_{\bar{\phi}}(x, y) \triangleq tr(\bar{\phi}^{\frac{1}{2}}x\bar{\phi}^{\frac{1}{2}}y^*)$ satisfies

$$s_{\bar{\phi}}|_{\pi(\mathcal{M})\times\pi(\mathcal{M})} \ = \ s_{\bar{\phi}|_{\pi(\mathcal{M})}} \ . \tag{4.6}$$

Now by a result of Haagerup and Størmer [10, Theorem 4.2], equation (4.6) and the equality $\pi(1) = s(\bar{\phi})$ imply the existence of a faithful normal conditional expectation $F : \pi(1)\mathcal{N}\pi(1) \to \pi(\mathcal{M})$ such that $\bar{\phi} = \bar{\phi} \circ F$. Defining E: $x \mapsto F(\pi(1)x\pi(1))$, $x \in \mathcal{N}$, we have

$$\bar{\phi} = \bar{\phi} \circ E = \phi \circ \pi^{-1} \circ E,$$

which establishes (4.5). □

Proof of Theorem 2. The main difference between Theorem 2 and Theorem 4.9 is that T is described on the spanning set of positive vectors $\{\varphi^{\frac{1}{p}} \mid \varphi \in \mathcal{M}_*^+\}$ instead of the dense set $\{\phi^{\frac{1}{p}}x \mid x \in \mathcal{M}\}$ (and T becomes everywhere-defined by linearity). The equation (1.2) is a noncommutative version of (1.1), so that T may be naturally viewed as a "noncommutative weighted composition operator". To establish (1.2), it is sufficient to show that the data π, E, w of Theorem 4.9 do not depend on the choice of $\phi \in \mathcal{M}_*^+$.

For this, choose an arbitrary faithful $\psi \in \mathcal{M}_*^+$ satisfying $\psi^{\frac{2}{p}} \leq C\phi^{\frac{2}{p}}$ for some $C < \infty$. (Again, this set is dense.) The assumption means that $d \triangleq (D\phi : D\psi)_{i/p}$ exists in $\mathcal{M}$, and by analytic continuation we have $\phi^{\frac{1}{p}} d = \psi^{\frac{1}{p}}$. (In fact the equation $d = \phi^{-\frac{1}{p}} \psi^{\frac{1}{p}}$ is justified rigorously in [40].) Cocycles are functorial with respect to normal *-isomorphisms, and they are invariant when weights are precomposed with a normal conditional expectation [44, Corollary IX.4.22], so

$$\pi(d) = (D(\phi \circ \pi^{-1} \circ E) : D(\psi \circ \pi^{-1} \circ E))_{i/p}.$$

Then we have

$$T(\psi^{\frac{1}{p}}) = T(\phi^{\frac{1}{p}} d) = w(\phi \circ \pi^{-1} \circ E)^{\frac{1}{p}} \pi(d) = w(\psi \circ \pi^{-1} \circ E)^{\frac{1}{p}}.$$

By density, this establishes (1.2). See also [38, Section 6].

Finally we note that the formulation of Theorem 2 does not require the σ-finiteness of $\mathcal{M}$. For each q in the net of σ-finite projections, the restricted isometry $T\colon L_p(q\mathcal{M}q) = qL_p(\mathcal{M})q \to L_p(\mathcal{N})$ is of the form (1.2) for some w_q, π_q, E_q. One only needs to glue them all together. This can be done in exactly the same way as in the proof of Theorem 3.1. □

5. Concluding remarks

Remark 1. If, in the setup of Theorem 2, we require that $w^*w = \pi(1)$ is the support of E, then π, w, and E are uniquely determined. For suppose that π_1, E_1, w_1 also define the 2-isometry T, and assume that $s(E_1) = \pi_1(1)$. Then for all $\varphi \in \mathcal{M}_*^+$,

$$w(\varphi \circ \pi^{-1} \circ E)^{\frac{1}{p}} = w_1(\varphi \circ \pi_1^{-1} \circ E_1)^{\frac{1}{p}} \Rightarrow \varphi \circ \pi^{-1} \circ E = \varphi \circ \pi_1^{-1} \circ E_1$$
$$\Rightarrow \pi^{-1} \circ E = \pi_1^{-1} \circ E_1 \Rightarrow \mathrm{id} = \pi^{-1} \circ E \circ \pi_1 \Rightarrow \pi = E \circ \pi_1.$$

Now take a projection $p \in \mathcal{M}$ and calculate

$$E(\pi(p)\pi_1(p)\pi(p)) = \pi(p)E(\pi_1(p))\pi(p) = \pi(p)^3 = \pi(p) = E(\pi(p)).$$

Since E is faithful on $\pi(1)\mathcal{N}\pi(1)$, we must have $\pi(p)\pi_1(p)\pi(p) = \pi(p)$, or $\pi_1(p) \geq \pi(p)$. Reversing the argument proves the equality of π and π_1, and the other data must be equal also (subject to $w^*w = \pi(1) = w_1^* w_1$).

Remark 2. Given such a 2-isometry T decomposed as in Theorem 2, the associated map

$$S\colon L_p(\mathcal{M}) \to L_p(\mathcal{N}), \qquad h \mapsto w^*T(h),$$

is a completely positive complete isometry whose range is completely positively and completely contractively complemented. If T is positive, then $w = \pi(1)$ and $T = S$ (and in particular T is already completely positive).

Remark 3. The arguments in Section 4 can be used to establish the following result, which seems to be of independent interest. The main step strengthens Lemma 4.8 by only requiring T to be an isometry (i.e. not a 2-isometry).

Proposition 5.1. *Let $\mathcal{M} \subset \mathcal{N}$ be a unital inclusion of von Neumann algebras with $\phi \in \mathcal{M}_*^+, \bar{\phi} \in \mathcal{N}_*^+$, both faithful, satisfying $\bar{\phi}\,|_{\mathcal{M}} = \phi$. Assume that the map*

$$T_p\colon L_p(\mathcal{M}) \to L_p(\mathcal{N}); \qquad \phi^{\frac{1}{p}} x \mapsto \bar{\phi}^{\frac{1}{p}} x, \qquad x \in \mathcal{M},$$

is isometric for some $1 \leq p \neq 2 < \infty$. Then there exists a faithful normal conditional expectation $E\colon \mathcal{N} \to \mathcal{M}$ such that $\bar{\phi} = \phi \circ E$.

Proof. It follows from Lemma 4.4, Corollary 4.5, and Proposition 4.7 that if T_p is isometric for one p in the given range, it is isometric for all. In particular, T_4 is isometric, so that for any $\mathcal{M} \ni y \geq 0$,

$$\|\phi^{\frac{1}{4}} y \phi^{\frac{1}{4}}\| = \|\phi^{\frac{1}{4}} y^{\frac{1}{2}}\|^2 = \|\bar{\phi}^{\frac{1}{4}} y^{\frac{1}{2}}\|^2 = \|\bar{\phi}^{\frac{1}{4}} y \bar{\phi}^{\frac{1}{4}}\|. \tag{5.1}$$

Suppose that $y \in \mathcal{M}$ is only self-adjoint. For all scalar $t \geq \|y\|$, (5.1) implies

$$\begin{aligned} \|\phi^{\frac{1}{4}} y \phi^{\frac{1}{4}}\|^2 + 2t\phi(y) + t^2\|\phi\| &= \|\phi^{\frac{1}{4}}(y + t1)\phi^{\frac{1}{4}}\|^2 \\ &= \|\bar{\phi}^{\frac{1}{4}}(y + t1)\bar{\phi}^{\frac{1}{4}}\|^2 \\ &= \|\bar{\phi}^{\frac{1}{4}} y \bar{\phi}^{\frac{1}{4}}\|^2 + 2t\bar{\phi}(y) + t^2\|\bar{\phi}\|. \end{aligned}$$

Since these are polynomials in t which agree on a half-line, they must have the same constant terms. We conclude that (5.1) holds for all self-adjoint $y \in \mathcal{M}$.

In other words the map $\phi^{\frac{1}{4}} y \phi^{\frac{1}{4}} \mapsto \bar{\phi}^{\frac{1}{4}} y \bar{\phi}^{\frac{1}{4}}$ (with $y \in \mathcal{M}_{sa}$) is isometric between the real Hilbert spaces $L_2(\mathcal{M})_{sa}$ and $L_2(\mathcal{N})_{sa}$. It therefore preserves inner products, which is the same as saying that the self-polar form s_ϕ agrees with $s_{\bar{\phi}}$ restricted to $\mathcal{M} \times \mathcal{M}$. Then the Haagerup-Størmer result [10, Theorem 4.2] again implies the existence of a faithful normal conditional expectation $E\colon \mathcal{N} \to \mathcal{M}$ satisfying $\bar{\phi} = \phi \circ E$. □

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